Problem Set 8

In this problem set, you'll transition away from the regular languages to the context-free languages and to the realm of Turing machines. This will be your first foray beyond the limits of what computers can over hope to accomplish, and we hope that you find this as exciting as we do!

As always, please feel free to drop by office hours or ask on Piazza if you have any questions. We'd be happy to help out.

Good luck, and have fun!

Due Friday, December 1st at 2:30PM.

Problem One: Designing CFGs

For each of the following languages, design a CFG for that language. ***Please use our online tool to design, test, and submit the CFGs in this problem***. To use it, visit the CS103 website and click the "CFG Editor" link under the "Resources" header. You should only have one member from each team submit your grammars; tell us who this person is when you submit the rest of the problems through GradeScope.

i. Given $\Sigma = \{$a, b, c$\}$, write a CFG for the language $\{ w \in \Sigma^* \mid w$ contains aa as a substring $\}$. For example, the strings aa, baac, and ccaabb are all in the language, but aba is not.

ii. Given $\Sigma = \{$a, b$\}$, write a CFG for the language $\{ w \in \Sigma^* \mid w$ is *not* a palindrome $\}$, the language of strings that are not the same when read forwards and backwards. For example, aab $\in L$ and baabab $\in L$, but aba $\notin L$, bb $\notin L$, and $\varepsilon \notin L$.

iii. Let Σ be an alphabet containing these symbols:

Ø ℕ { } , ∪

We can form strings from these symbols which represent sets. Here's some examples:

Ø	{Ø, ℕ} ∪ ℕ ∪ Ø	{Ø} ∪ ℕ ∪ {ℕ}	{Ø, Ø}
{{ℕ, Ø} ∪ {Ø}}	ℕ ∪ {ℕ, Ø}	{}	{ℕ}
{Ø, {Ø, {Ø}}}	{{{{ℕ}}}}	ℕ	{Ø, {}}

Notice that some of these sets, like {Ø, Ø} are syntactically valid but redundant, and others like {} are syntactically valid but not the cleanest way of writing things. Here's some examples of strings that don't represent sets or aren't syntactically valid:

ε	}Ø{	Ø{ℕ}	{{}
ℕ, Ø, {Ø}	{, ℕ}	{ ℕ Ø },	{,}
{Ø	}} ℕ	{ Ø, Ø, Ø, }	{ℕ, , , Ø}

Write a CFG for the language $\{ w \in \Sigma^* \mid w$ is a syntactically valid string representing a set $\}$. When using the CFG tool, please use the letters n, u, and o in place of ℕ, ∪, and Ø, respectively.

Fun fact: The starter files for Problem Set One contain a parser that's designed to take as input a string representing a set and to reconstruct what set that is. The logic we wrote to do that parsing was based on a CFG we wrote for sets and set theory. Take CS143 if you're curious how to go from a grammar to a parser!

Problem Two: The Complexity of Addition

This problem explores the following question:

How hard is it to add two numbers?

Suppose that we want to check whether $x + y = z$, where x, y, and z are all natural numbers. If we want to phrase this as a problem as a question of strings and languages, we will need to find some way to standardize our notation. In this problem, we will be using the ***unary number system***, a number system in which the number n is represented by writing out n `1`'s. For example, the number 5 would be written as `11111`, the number 7 as `1111111`, and the number 12 as `111111111111`.

Given the alphabet $\Sigma = \{$`1`, `+`, `=`$\}$, we can consider strings encoding $x + y = z$ by writing out x, y, and z in unary. For example:

$4 + 3 = 7$ would be encoded as `111+1111=1111111`

$7 + 1 = 8$ would be encoded as `1111111+1=11111111`

$0 + 1 = 1$ would be encoded as `+1=1`

Consider the alphabet $\Sigma = \{$`1`, `+`, `=`$\}$ and the following language, which we'll call *ADD*:

$$\{ \texttt{1}^m\texttt{+}\texttt{1}^n\texttt{=}\texttt{1}^{m+n} \mid m, n \in \mathbb{N} \}$$

For example, the strings `111+1=1111` and `+1=1` are in the language, but `1+11=11` is not, nor is the string `1+1+1=111`.

i. Prove or disprove: the language *ADD* defined above is regular.

ii. Write a context-free grammar for *ADD*, showing that *ADD* is context-free. (Please submit your CFG online.)

Problem Three: The Complexity of Pet Ownership

This problem explores the following question:

How hard is it to walk your dog without a leash?

Let's imagine that you're going for a walk with your dog, but this time don't have a leash. As in Problem Set Six and Problem Set Seven, let $\Sigma = \{$`y`, `d`$\}$, where `y` means that you take a step forward and `d` means that your dog takes a step forward. A string in Σ^* can be thought of as a series of events in which either you or your dog moves forward one unit. For example, the string `yydd` means that you take two steps forward, then your dog takes two steps forward.

Let *DOGWALK* = $\{ w \in \Sigma^* \mid w$ describes a series of steps where you and your dog arrive at the same point $\}$. For example, the strings `yyyddd`, `ydyd`, and `yyydddddyyy` are all in *DOGWALK*.

i. Prove or disprove: the language *DOGWALK* defined above is regular.

ii. Write a context-free grammar for *DOGWALK*, showing that *DOGWALK* is context-free. (Please submit your CFG online.)

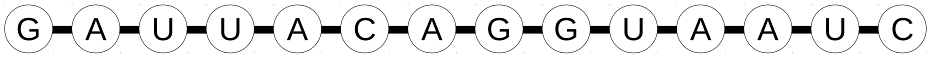
Problem Four: RNA Hairpins

RNA strands consist of strings of *nucleotides*, molecules which encode genetic information. Computational biologists typically represent each RNA strand as a string made from four different letters, A, C, G, and U, each of which represents one of the four possible nucleotides.

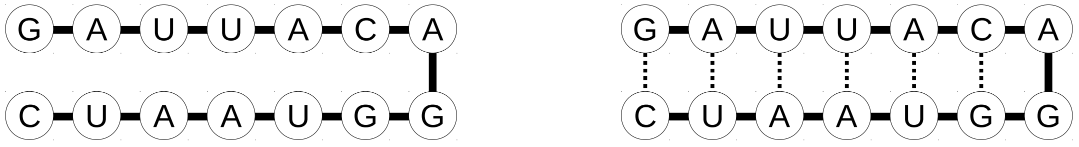
Each of the the four nucleotides has an affinity for a specific other nucleotide. Specifically:

A has an affinity for U (and vice-versa) C has an affinity for G (and vice-versa)

This can cause RNA strands to fold over and bind with themselves. Consider this RNA strand:



If you perfectly fold this RNA strand in half, you get the following:



Notice that each pair of nucleotides – except for the A and the G on the far right – are attracted to the corresponding nucleotide on the other side of the RNA strand. Because of the natural affinities of the nucleotides in the RNA strand, the RNA strand will be held in this shape. This is an example of an *RNA hairpin*, a structure with important biological roles.

For the purposes of this problem, we'll say that an RNA strand forms a hairpin if

- it has even length (so that it can be cleanly folded in half);
- it has length at least ten (there are at least four pairs holding the hairpin shut); and
- all of its nucleotides, except for the middle two, have an affinity for its corresponding nucleotide when folded over. (The middle two nucleotides in a hairpin might coincidentally have an affinity for one another, but it's not required. For example, CCCC$\underline{\text{AU}}$GGGG forms a hairpin.)

This problem explores the question

How hard is it to determine whether an RNA strand forms a hairpin?

Let $\Sigma = \{$A, C, G, U$\}$ and let $L_{RNA} = \{ w \in \Sigma^* \mid w$ represents an RNA strand that forms a hairpin $\}$. For example, the strings UGACCCGUCA, GUACAAGUAC, UUUUUUUUUAAAAAAAAA, and CCAACCUUGG are all in L_{RNA}, but the strings AU, AAAAACUUUUU, GGGC, and GUUUUAAAAG are all not in L_{RNA}.

i. Prove that L_{RNA} is not regular. Since this language imposes a lot of requirements on the strings it contains, if in the course of your proof you want to claim that a particular string is or is not in L_{RNA}, please articulate clearly why the string does or does not meet each of the requirements of strings in L_{RNA}.

ii. Design a CFG for L_{RNA}, which proves that the language is context-free. Please submit your grammar online.

Problem Five: Right-Linear Grammars

A context-free grammar is called a ***right-linear grammar*** if every production in the grammar has one of the following three forms:

- A → ε
- A → B, where B is a nonterminal.
- A → aB, where a is a terminal and B is a nonterminal.

For example, the following is a right-linear grammar:

A → aB | bB | ε

B → aC | bA | C

C → bA | aA | ε

The right-linear grammars are all context-free grammars, so their languages are all context-free. However, it turns out that this class of grammars precisely describe the regular languages. That is, a language *L* is regular if and only if there is a right-linear grammar *G* such that $L = \mathscr{L}(G)$.

i. Let *G* be a right-linear grammar. Describe how to construct an NFA *N* such that $\mathscr{L}(G) = \mathscr{L}(N)$. You don't need to formally prove that your construction is correct, but you should give a good intuitive explanation as to why the grammar has the same language as the generated NFA. Additionally, to illustrate your construction, show the NFA you'd construct from the following right-linear grammar:

   A → aB | bC

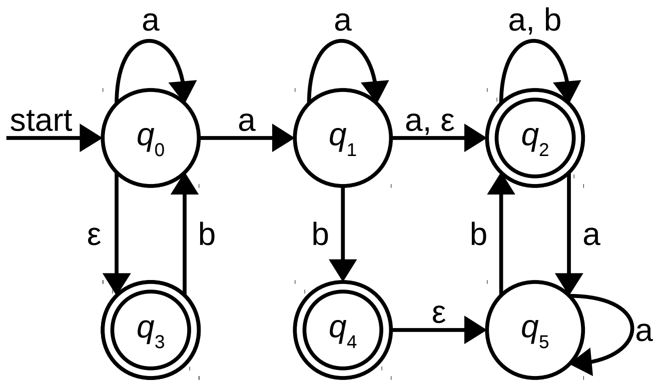
   B → aB | ε

   C → aD | A | bC

   D → aD | bD | ε

   Please submit your NFA for this question through our DFA/NFA editor.

ii. Let *N* be an NFA. Describe how to construct a right-linear grammar *G* such that $\mathscr{L}(G) = \mathscr{L}(N)$. You don't need to formally prove that your construction is correct, but you should give a good intuitive explanation as to why the generated grammar has the same language as the NFA. Additionally, to illustrate your construction, show the grammar that you'd construct from the following NFA:



   Please submit your right-linear grammar through our online CFG tool.

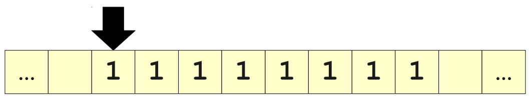
Problem Six: The Collatz Conjecture

The *Collatz conjecture* is a famous conjecture (an unproved claim) that says the following procedure (called the *hailstone sequence*) terminates for all positive natural numbers n:

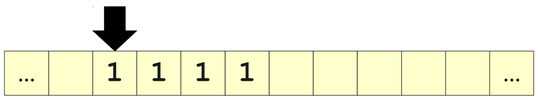
- If $n = 1$, stop.
- If n is even, set $n = n / 2$ and repeat from the top.
- If n is odd, set $n = 3n + 1$ and repeat from the top.

Let $L = \{ \texttt{1}^n \mid n \geq 1$ and the hailstone sequence terminates for $n \}$ be a language over the singleton alphabet $\Sigma = \{\texttt{1}\}$. It turns out that it's possible to build a TM for this language (using the terminology from our upcoming lecture on Monday, we'd say $L \in \mathbf{RE}$), and in this problem you'll do just that. Parts (i) and (ii) will ask you to design two useful subroutines, and you'll assemble the overall machine in part (iii).

i. Design a TM subroutine that, given a tape holding a string of the form $\texttt{1}^{2n}$ (where $n \in \mathbb{N}$) surrounded by infinitely many blanks, ends with $\texttt{1}^n$ written on the tape, surrounded by infinitely many blanks. You can assume the tape head begins reading the first `1` (or points to an arbitrary blank cell in the case where $n = 0$), and your TM should end with the tape head reading the first `1` of the result (or any blank cell if $n = 0$). For example, given the initial configuration

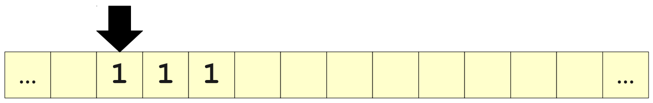


your TM subroutine would end with this configuration:

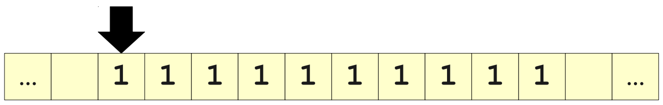


You can assume that there are an even number of `1`s on the tape at startup and can have your TM behave however you'd like if this isn't the case. Please use our provided TM editor to design, develop, test, and submit your answer to this question. Since our TM tool doesn't directly support subroutines, just have your machine accept when it's done. *(For reference, our solution has fewer than 10 states. If you have significantly more than this, you might want to change your approach.)*

ii. Design a TM subroutine that, given a tape holding a string of the form $\texttt{1}^n$ (for some $n \in \mathbb{N}$), surrounded by infinitely many blanks, ends with $\texttt{1}^{3n+1}$ written on the tape, surrounded by infinitely many blanks. You can assume that the tape head begins reading the first `1`, and your TM should end with the tape head reading the first `1` of the result. For example, given this configuration



your TM subroutine would end with this configuration:



You can assume the number of `1`s on the tape at startup is odd and can have your TM behave however you'd like if this isn't the case. Please use our provided TM editor to design, develop, test, and submit your answer to this question. Since our TM tool doesn't directly support subroutines, just have your machine accept when it's done. *(For reference, our solution has fewer than 10 states. If you have significantly more than this, you might want to change your approach.)*

iii. Draw the state transition diagram for a Turing machine *M* that recognizes *L*. Our TM tool is configured for this problem so that you can use our reference solutions for parts (i) and (ii) as subroutines in your solution. To do so, follow these directions:

1. Create states named `half`, `half_`, `trip`, and `trip_`.
2. To execute the subroutine that converts $\texttt{1}^{2n}$ into $\texttt{1}^n$, transition into the state named `half`. When that subroutine finishes, the TM will automagically jump into the state labeled `half_`. You do not need to – and should not – define any transitions into `half_` or out of `half`.
3. To execute the subroutine that converts $\texttt{1}^n$ into $\texttt{1}^{3n+1}$, transition into the state named `trip`. When that subroutine finishes, the TM will automagically jump into the state labeled `trip_`. You do not need to – and should not – define any transitions into `trip_` or out of `trip`.

Calling ahead to Monday's lecture: a TM *M* recognizes a language *L* if *M* accepts all of the strings in *L* and either rejects or loops on all strings that are not in *L*. In other words, your TM should accept every string in *L*, and for any string not in *L* it can either loop infinitely or reject the string.

Please use our provided TM editor to design, develop, test, and submit your answer to this question. *(For reference, our solution has fewer than 15 states. If you have significantly more than this, you might want to change your approach.)*

Problem Seven: Jumbled Jargon

(We will cover the material necessary to solve this problem on Monday, when we get back from break.)

We've introduced a number of terms and definitions pertaining to Turing machines, languages, and what it means to solve a problem. Some of the terms we've described are adjectives that can only describe TMs, while others are adjectives that can only describe languages. Using them incorrectly leads to statements that aren't mathematically meaningful.

To reason by analogy, consider the statement "the set ℕ is even." This statement isn't meaningful, because "even" can only be applied to individual natural numbers, and ℕ isn't a natural number. Similarly, the statement $1 \in 5$ isn't meaningful, since 5 isn't a set. The statement $\mathbb{Z} \subseteq \mathbb{N}$ is meaningful but not true – it's the mathematical equivalent of a grammatically correct statement that just happens to be false.

Below is a series of statements. For each statement, decide whether that statement is mathematically meaningful or not. If it's not mathematically meaningful, explain why not. If it is mathematically meaningful, determine whether it's true or false and briefly justify your answer.

i. If *M* is a Turing machine, *w* is a string, and *M* accepts *w*, then A_{TM} accepts $\langle M, w \rangle$.

ii. If *M* is a Turing machine, *w* is a string, and *M* loops on *w*, then $\langle M, w \rangle \notin \mathscr{L}(\text{U}_{\text{TM}})$.

iii. U_{TM} is decidable.

iv. $\langle \text{U}_{\text{TM}} \rangle$ is decidable.

v. $\{\langle \text{U}_{\text{TM}} \rangle\}$ is decidable.

Problem Eight: What Does it Mean to Solve a Problem?

(We will cover the material necessary to solve this problem on Monday, when we get back from break.)

Let L be a language over Σ and M be a TM with input alphabet Σ. Below are three properties that may hold for M:

1. M halts on all inputs.
2. For any string $w \in \Sigma^*$, if M accepts w, then $w \in L$.
3. For any string $w \in \Sigma^*$, if M rejects w, then $w \notin L$.

At some level, for a TM to claim to solve a problem, it should have at least some of these properties. Interestingly, though, just having two of these properties doesn't say much.

i. Prove that if L is any language over Σ, then there is a TM M that satisfies properties (1) and (2).

ii. Prove that if L is any language over Σ, then there is a TM M that satisfies properties (1) and (3).

iii. Prove that if L is any language over Σ, then there is a TM M that satisfies properties (2) and (3).

iv. Suppose that L is a language over Σ for which there is a TM M that satisfies properties (1), (2), and (3). What can you say about L? Prove it.

Problem Nine: R and RE Languages

(We will cover the material necessary to solve this problem on Monday, when we get back from break.)

The following problems are designed to explore some of the nuances of how Turing machines, languages, decidability, and recognizability all relate to one another. We hope that by working through them, you'll get a much better understanding of key computability concepts.

i. Give a high-level description of a TM M such that $\mathscr{L}(M) \in \mathbf{R}$, but M is not a decider (you can draw a concrete example of TM, or give pseudocode for a program along the lines of what we've done in class). Briefly justify your answer. This shows that just because a TM's language is decidable, it's not necessarily the case that the TM itself must be a decider.

ii. Only *languages* can be decidable or recognizable; there's no such thing as an "undecidable string" or "unrecognizable string." Prove that for every string w, there's an **R** language containing w and an **RE** language containing w.

iii. Here's a weird one. Let Σ be an alphabet containing all characters that can appear in a person's name. Prove that the following language is decidable, subject to the assumption that there is a single US presidential election in 2020 and that it ends with a single winner:

$$\{ \ w \in \Sigma^* \mid w \text{ is the name of the winner of the 2020 presidential election } \}$$

Optional Fun Problem: TMs and Regular Languages (1 Point Extra Credit)

Here's a really interesting connection between TMs and DFAs. Imagine you have a TM M with the following property: there exists some natural number k such that after M is started up on any string w, M always halts (accepts or rejects) after at most k steps. Prove that $\mathscr{L}(M)$ is regular, where $\mathscr{L}(M)$ is the set of all strings that M accepts.